# ON THE OPTIMAL STABILIZATION OF NONLINEAR SYSTEMS

## (OB OPTIMAL'NOI STABILIZATSII NELINEINYKH SISTEM)

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The author considers the problem of the formation of the guidance action in a nonlinear control system under the condition of the minimum of the integral for the estimate of quality for small initial disturbances. The problem is solved by the methods of Liapunov and Chetaev in stability theory.

**1.** Let us consider the system of equations of a disturbed motion

$$\frac{dx_i}{dt} = f_i(x, u) \qquad (i = 1, \ldots, n) \tag{1.1}$$

Here $x = \{x_1, \ldots, x_n\}$ is an $n$-dimensional vector in the phase coordinates of the system, $u$ is a scalar function of the $x_i$ coordinates which describes the guidance action, the control.

It is required to find such a control $u(x)$ that for $x = 0$ the undisturbed motion is asymptotically stable, and for which along the trajectories of the system the following integral will have a minimum:

$$\int_0^\infty G(x, u)\, dt = \min \tag{1.2}$$

Here $G(x, u)$ is a given function which characterizes the criterion of quality. Letov [1,2] has studied this type of problem for a linear system under the condition when $G(x, u)$ is a quadratic form.

We shall assume that the functions $f_i(x, u)$ and $G(x, u)$ are analytic functions in some neighborhood of the origin $x = 0$, $u = 0$, and that they can be expanded in convergent power series

$$f_i(x, u) = \sum_{m=1}^{\infty} \varphi_i^{(m)}(x) + \sum_{k=1}^{\infty} A_{ik} u^k + \sum_{k,\, m=1}^{\infty} \varphi_{ik}^{(m)}(x)\, u^k \qquad (i = 1, \ldots, n) \tag{1.3}$$

$$G(x, u) = \sum_{m=2}^{\infty} \psi^{(m)}(x) + \sum_{k=2}^{\infty} B_k u^k + \sum_{k,\, m=1}^{\infty} \psi_k^{(m)}(x) u^k \qquad (B_2 \neq 0) \tag{1.4}$$

where the symbol $(m)$ gives the order of the form.

We shall give a sufficient criterion for the optimum of the control. This criterion will be based on the ideas of the method of Liapunov's function [3] which involves certain ideas from the theory of dynamic programming [4]. For this purpose we consider two functions $v(x)$ and $u^0(x)$ satisfying the following conditions:

a) the function $v(x)$ satisfies the conditions of Liapunov's theorem on asymptotic stability [5, p. 29];

b) the derivative function of $v(x)$ satisfies, in view of the system (1.1), with $u = \xi(x)$, the equation

$$(dv\,/\,dt)_{\xi(x)} = -G(x, \xi(x))$$

c) the function

$$H(x, \xi) = (dv\,/\,dt)_{\xi(x)} + G(x, \xi)$$

has a minimum at each point $x$ of some neighborhood of the origin if one sets $\xi = u^\circ(x)$.

*Theorem 1.1.* If one can find functions $v(x)$ and $u^\circ(x)$ satisfying conditions (a) to (c), then the control $u = u^\circ(x)$ will be an optimal control, i.e. it will satisfy the condition (1.2)*.

*Proof.* From the condition (b) it follows by integration with respect to $t$ that along the trajectories of the system (1.1), with $\xi = u^\circ(x)$

$$v(x(t)) - v(x_0) = \int_0^t G(x(\tau), u^\circ(x(\tau)))\, d\tau$$

Let us now take the limit as $t \to \infty$, taking into account the fact that, by condition (a), $x \to 0$ when $t \to \infty$. We then obtain

$$v(x_0) = \int_0^\infty G(x(t), u^\circ(x(t)))\, dt$$

Let us assume that the theorem is false, that the control $u^\circ(x)$ satisfying (a) to (c) is not an optimal control, namely, that there exists a

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* Such a criterion was given for linear systems in [6].

control $u_1(x)$ such that

$$\int_0^\infty G\,(x\,(t),\ u_1\,(x\,(t)))\,dt < \int_0^\infty G\,(x\,(t),\ u^\circ\,(x\,(t)))\,dt$$

for some initial condition $x_0$. From condition (c) we have

$$(dv\,/\,dt)_{u_1(x)} \geqslant -G\,(x,\ u_1\,(x))$$

Integrating this inequality along a trajectory of the system (1.1) we obtain

$$v\,(x_0) \leqslant \int_0^\infty G\,(x\,(t),\ u_1\,(x\,(t)))\,dt$$

which contradicts the hypothesis. Hence, the theorem is proved.

**2.** In this section we present a formal method for the construction of a Liapunov function $v(x)$, and of an optimal control $u^\circ(x)$ in the form of a power series in $x$. For the sake of simplicity, we shall drop the index $^\circ$ on $u^\circ(x)$.

From Theorem 1.1 it follows that it is sufficient to find functions $v(x)$ and $u(x)$ satisfying the condition

$$\min_u \left\{\frac{dv}{dt} + G\,(x,\ u)\right\} = 0 \tag{2.1}$$

This condition yields the system of equations

$$\sum_{i=1}^n f_i\,(x,\ u)\,\frac{\partial v}{\partial x_i} + G\,(x,\ u) = 0 \tag{2.2}$$

$$\sum_{i=1}^n \frac{\partial f_i}{\partial u}\,\frac{\partial v}{\partial x_i} + \frac{\partial G}{\partial u} = 0 \tag{2.3}$$

We shall restrict ourselves at first to the lowest-degree terms in $x$ and $u$ of the series (1.3) and (1.4). Then the functions $f_i(x,\ u) \approx \phi_i^{(1)}(x) + A_{i1}u$ $(i = 1,\ \ldots,\ n)$ will be linear functions, while $G(x,u) \approx \psi^{(2)}(x) + \psi^{(1)}(x)u + B_2u^2$ must be considered to be a quadratic form (positive-definite).* The system of equations (1.1) will be a linear one

* From practical considerations it follows that such functions estimate especially well the quality of transient processes [1,2].

$$\frac{dx_i}{dt} = \varphi_i^{(1)}(x) + A_{i1}u^{(1)}(x) \qquad (i = 1, \ldots, n) \tag{2.4}$$

and the condition for an optimum of the equation takes the form

$$\int_0^\infty (\psi^{(2)}(x) + \psi_1^{(1)}(x)\, u^{(1)} + B_2 (u^{(1)})^2)\, dt = \min \tag{2.5}$$

From Equation (2.3) we find the control to be

$$u^{(1)}(x) = -\frac{1}{2B_2}\sum_{i=1}^{n} A_{i1}\frac{\partial v^{(2)}}{\partial x_i} - \frac{1}{2B_2}\,\psi_1^{(1)}(x) \tag{2.6}$$

The Liapunov function $v^{(2)}(x)$ will be a quadratic form whose coefficients satisfy a system of quadratic equations which are solvable under certain conditions. These conditions were found in [6]. We shall assume that the conditions for solvability of the linear system are satisfied, and that we know the solution of the linear problem $v^{(2)}(x)$ and $u^{(1)}(x)$.

The solutions of Equations (2.2) and (2.3) are to be found for the general case in the form

$$v(x) = v^{(2)}(x) + v^{(3)}(x) + \ldots + v^{(m)}(x) + \ldots \tag{2.7}$$

$$u(x) = u^{(1)}(x) + u^{(2)}(x) + \ldots + u^{(m-1)}(x) + \ldots \tag{2.8}$$

Let us formally substitute (1.3), (1.4), (2.7) and (2.8) into the system of equations (2.2) and (2.3), and let us equate to zero the coefficients of the powers of $x$. Hereby, the terms of the $m$th order in Equation (2.2) will correspond to the terms of the $(m-1)$st order in (2.3). For the functions $v^{(2)}(x)$ and $u^{(1)}(x)$ one obtains the same equation as above in the consideration of the linear problem (2.4), (2.5).

Let us assume that the functions $v^{(3)}(x), \ldots, v^{(m-1)}(x)$, and $u^{(2)}(x), \ldots, u^{(m-2)}(x)$ have been found. Let us write down the $m$th-order terms of Equation (2.2), and the terms of the $(m-1)$st order of Equation (2.3). We thus obtain

$$\sum_{i=1}^{n} \varphi_i^{(1)}(x)\frac{\partial v^{(m)}}{\partial x_i} + u^{(m-1)}(x)\sum_{i=1}^{n} A_{i1}\frac{\partial v^{(2)}}{\partial x_i} + u^{(1)}(x)\sum_{i=1}^{n} A_{i1}\frac{\partial v^{(m)}}{\partial x_i} +$$
$$+ 2B_2 u^{(1)}(x)\, u^{(m-1)}(x) + u^{(m-1)}(x)\,\psi_1^{(1)}(x) = F_1^{(m)}(x) \tag{2.9}$$

$$\sum_{i=1}^{n} A_{i1}\frac{\partial v^{(m)}}{\partial x_i} + 2B_2 u^{(m-1)}(x) = F_2^{(m-1)}(x) \tag{2.10}$$

Taking into account (2.4) and (2.6), we can transform Equation (2.9) to the form

$$\left(\frac{dv^{(m)}}{dt}\right)_{(2.4)} = F_1^{(m)}(x) \tag{2.11}$$

Here $F_1^{(m)}(x)$, $F_2^{(m-1)}(x)$ are known forms; the symbol $(dv^{(m)}/dt)_{(2.4)}$ indicates the derivative on the basis of (2.4) when $u^{(1)}(x)$ is given by (2.6).

Since the system (2.4) is asymptotically stable, we have by Liapunov's theorem [7, p. 61] that there exists a unique solution of Equation (2.11). Knowing $v^{(m)}(x)$, we can find $u^{(m-1)}(x)$ with the aid of (2.10). Thus one can determine successively the forms of any order in the series (2.7) and (2.8). Hence, if the problem (2.5) can be solved for the linear approximation, then there exists a unique formal solution of the non-linear problem.

**3.** In this section we prove, for a typical case, the convergence of the formal series whose construction was described in Section 2.

Let us consider the system given by the equations

$$\frac{dx_i}{dt} = f_i(x) + b_i u \qquad \left(f_i(x) = \sum_{m=1}^{\infty} f_i^{(m)}(x)\right) \qquad (i = 1, \ldots, n) \tag{3.1}$$

where $f_i(x)$ are analytic functions.

Suppose that along the trajectories of the system (3.1) the following integral is minimized:

$$\int_0^\infty \left(\sum_{i=1}^n x_i^2 + u^2\right) dt = \min \tag{3.2}$$

The functions $v(x)$ and $u(x)$ satisfy the conditions

$$\sum_{i=1}^n f_i(x) \frac{\partial v}{\partial x_i} - \frac{1}{4}\left(\sum_{i=1}^n b_i \frac{\partial v}{\partial x_i}\right)^2 + \sum_{i=1}^n x_i^2 = 0 \tag{3.3}$$

$$u = -\frac{1}{2} \sum_{i=1}^n b_i \frac{\partial v}{\partial x_i} \tag{3.4}$$

We shall assume that the solution of the linear problem $v^{(2)}(x)$ and $u^{(1)}(x)$ is known, and that the linear system of the first approximation

$$\frac{dx_i}{dt} = f_i^{(1)}(x) - \frac{1}{2} b_i \sum_{i=1}^n b_i \frac{\partial v^{(2)}}{\partial x_i} \qquad (i = 1, \ldots, n) \tag{3.5}$$

is asymptotically stable. Let

$$x_i = c_{i1} y_1 + \ldots + c_{in} y_n \qquad (i = 1, \ldots, n) \tag{3.6}$$

be a linear nonsingular transformation [8] which reduces the quadratic form $v^{(2)}(x)$ to the form

$$v^{(2)}(y) = y_1^2 + \ldots + y_n^2 \tag{3.7}$$

Let its inverse transformation be given by

$$y_i = d_{i1}x_1 + \ldots + d_{in}x_n \quad (i = 1, \ldots, n) \tag{3.8}$$

Since the roots of the characteristic equation are invariant under any linear nonsingular transformation, the linear system in the new variables will also be asymptotically stable, and it will have the form

$$\frac{dy_j}{dt} = \sum_{i=1}^{n} d_{ji} f_i^{(1)}(x(y)) - \frac{1}{2} B_j \sum_{j=1}^{n} B_j \frac{\partial v^{(2)}}{\partial y_j} \qquad (j = 1, \ldots, n) \tag{3.9}$$

while Equation (3.3) will become

$$\sum_{j=1}^{n} F_j(y) \frac{\partial v}{\partial y_j} - \frac{1}{4}\Big(\sum_{j=1}^{n} B_j \frac{\partial v}{\partial y_j}\Big)^2 + \sum_{ij=1}^{n} B_{ij} y_i y_j = 0 \tag{3.10}$$

where

$$F_j(y) = \sum_{i=1}^{n} d_{ji} f_i(x(y)), \; B_j = \sum_{i=1}^{n} d_{ji} b_i \qquad (j = 1, \ldots, n) \tag{3.11}$$

The functions $F_j(y)$ will obviously be analytic functions in the new variables, and Equation (3.10) will be of the same type as (3.3). Hence we shall assume that the transformation (3.6) was made to begin with, and that the solution of the problem (3.5) has the form $v^{(2)}(x) = x_1^2 + \ldots + x_n^2$.

For what follows we need the next assertion.

If $f_1(x), \ldots, f_n(x)$ is the system of analytic functions of the right-hand sides of Equations (3.1), then there exists a convergent power series

$$\sum_{m=1}^{\infty} C_m r^m \qquad \Big(r^2 = \sum_{i=1}^{n} x_i^2\Big)$$

such that the following inequality is valid:

$$|f_i^{(m)}(x)| \leqslant C_m r^m \qquad (i = 1, \ldots, n;\; m = 1, 2, 3 \ldots) \tag{3.12}$$

Let us substitute (2.7) and (3.1) into (3.3), and equate to zero the coefficients of the powers of $x$. Then we obtain the following set of equations for the determination of the terms of the series (2.7):

$$\left(\frac{dv^{(3)}}{dt}\right)_{(3.5)} = -\sum_{i=1}^{n} f_i^{(2)}(x)\frac{\partial v^{(2)}}{\partial x_i} \tag{3.13}$$

$$\left(\frac{dv^{(4)}}{dt}\right)_{(3.5)} = -\sum_{i=1}^{n} f_i^{(2)}(x)\frac{\partial v^{(3)}}{\partial x_i} - \sum_{i=1}^{n} f_i^{(3)}(x)\frac{\partial v^{(2)}}{\partial x_i} + \frac{1}{4}\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(3)}}{\partial x_i}\right)^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(\frac{dv^{(m)}}{dt}\right)_{(3.5)} = -\sum_{i=1}^{n} f_i^{(2)}(x)\frac{\partial v^{(m-1)}}{\partial x_i} - \sum_{i=1}^{n} f_i^{(3)}(x)\frac{\partial v^{(m-2)}}{\partial x_i} - \ldots$$
$$\ldots - \sum_{i=1}^{n} f_i^{(m-1)}(x)\frac{\partial v^{(2)}}{\partial x_i} + \frac{1}{2}\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(3)}}{\partial x_i}\right)\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(m-1)}}{\partial x_i}\right) +$$
$$+ \frac{1}{2}\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(4)}}{\partial x_i}\right)\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(m-2)}}{\partial x_i}\right) + \ldots + \frac{1}{4}\left(\sum_{i=1}^{n} b_i \frac{\partial v^{(m/2+1)}}{\partial x_i}\right)^2$$

where the last term occurs only when $m$ is even.

For the proof of the convergence of the series (2.7) we make use of the known inequalities

$$|v^{(m)}(x)| \leqslant A_m r^m, \qquad \left|\frac{\partial v^{(m)}}{\partial x_i}\right| \leqslant cmA_m r^{m-1} \tag{3.14}$$

where $c$ is a coefficient of proportionality which does not depend on the order of the form. We also utilize (3.12) and consider the series

$$v(r) = r^2 + A_3 r^3 + \ldots + A_m r^m + \ldots \tag{3.15}$$

The series (3.15) is a dominating series for (2.7). It is only necessary to establish estimates of the coefficients $A_m$ for which the series will converge. With the aid of the relations (3.13) we estimate $A_3$, $A_4$, $\ldots$, $A_m$, $\ldots$ . From the first equation of (3.13) we have

$$|v^{(3)}(x)| = \left|-\int_0^\infty \sum_{i=1}^{n} f_i^{(2)}(x)\frac{\partial v^{(2)}}{\partial x_i}\,dt\right| \leqslant 2ncC_2\int_0^\infty r^3 dt$$

Let us introduce the notation $n_1 = nc$, and make use of the inequality $r(t) \leqslant r_0 e^{-\alpha(t)}$ which is satisfied by the solution of the system (3.5) because $r^2 = x_1^2 + \ldots + x_n^2 = v^{(2)}(x)$ is a Liapunov function for the system of the first approximation.

Then we obtain

$$|v^{(3)}(x)| \leqslant \frac{1}{3\alpha}2n_1C_2r_0^3, \qquad \text{or} \qquad A_3 = \frac{1}{3\alpha}2n_1C_2$$

In exactly the same way we estimate $v^{(4)}(x)$, $v^{(5)}(x)$, and so on. We thus obtain the following values for the coefficients of the series (3.15):

$$A_3 = \frac{1}{3\alpha} 2n_1C_2$$

$$A_4 = \frac{1}{4\alpha}\left[3n_1C_2A_3 + 2n_1C_3 + \frac{9}{4} n_1{}^2b^2 A_3{}^2\right] \tag{3.16}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_m = \frac{1}{m\alpha}\Big[(m-1)\, n_1C_2A_{m-1} + \ldots + 2n_1C_{m-1} +$$

$$+\frac{1}{2} n_1{}^2b^2 3\,(m-1)\, A_3A_{m-1} + \ldots + \frac{1}{4}\, n_1{}^2b^2 \left(\frac{m}{2}+1\right)^2 A^2_{\left(\frac{m}{2}+1\right)}\Big]$$

$$(b = \max\{|b_1|, \ldots, |b_n|\})$$

For the proof of the convergence of the series (3.15) we make use of the method of the dominating series, i.e. we construct a convergent series with positive coefficients

$$v_1(r) = B_2r^2 + B_3r^3 + \ldots + B_mr^m + \ldots \tag{3.17}$$

such that the inequality [9]

$$A_m \leqslant B_m \tag{3.18}$$

will be satisfied for all $m$ after a certain one.

Let us consider the solution of the equation

$$n_1\left(-a_1 + \sum_{m=2}^{\infty} C_m\mu^m\right) r\,\frac{dv_2}{dr} + n_1{}^2b^2\left(\frac{dv_2}{dr}\right)^2 + \alpha_1 r^2 = 0 \tag{3.19}$$

Here $\mu$ is some parameter for which the convergence of the series $C_2\mu^2 + C_3\mu^3 + \ldots + C_m\mu^m \ldots$ is not violated; the numbers $C_m$ satisfy the inequalities (3.12), while the numbers $a_1$ and $\alpha_1$ satisfy the inequality $\alpha_1 < a_1{}^2/4b^2$.

The function $v_2(r)$, which is a solution of Equation (3.19), has the form

$$v_2(r) = \frac{1}{2} B(\mu)\, r^2 \tag{3.20}$$

where $B(\mu)$, a solution of a quadratic equation, is an analytic function of the parameter $\mu$, and can be represented in the form of the series

$$B(\mu) = \sum_{m=2}^{\infty} B_m\mu^{m-2} \tag{3.21}$$

Let us substitute (3.20) and (3.21) into (3.19), and equate to zero the coefficients of the various powers of $x$. The coefficient of the first term of the series (3.21) satisfies the quadratic equation

$$B_2^2 - \frac{a_1}{n_1 b^2} B_2 + \frac{\alpha_1}{n_1^2 b^2} = 0 \tag{3.22}$$

Equation (3.22) has two real positive roots, and the solution is given by the expression

$$B_2 = \frac{a_1}{2n_1 b^2} - \frac{1}{n_1 b} \sqrt{\frac{a_1^2}{4b^2} - \alpha_1} \tag{3.23}$$

The coefficient of the second term of the series is determined by means of the equation

$$(n_1 a_1 - 2n_1^2 b^2 B_2)\, B_3 = n_1 C_2 B_2 \tag{3.24}$$

The expression in the parentheses will be the same in all equations for the determination of the coefficients of the series (3.21). It is obvious that

$$n_1 a_1 - 2n_1^2 b^2 B_2 = 2n_1 b \sqrt{\frac{a_1^2}{4b^2} - \alpha_1} \tag{3.25}$$

Let us assume that $a_1$ is known, and we are to select $a_1$ so that the following equation be satisfied:

$$\frac{a_1^2}{4b^2} - \alpha_1 = \frac{\alpha^2}{4n_1^2 b^2} \tag{3.26}$$

Then we will have

$$n_1 a_1 - 2n_1^2 b^2 B_2 = \alpha, \quad \text{or} \quad B_2 = \frac{1}{2n_1 b^2}\left(a_1 - \frac{\alpha}{n_1}\right) \tag{3.27}$$

Now let us assume that $a_1 \geqslant 2n_1 b^2 + \alpha/n_1$; then $B_2 \geqslant 1$. Carrying out the computation, we obtain the following relations for the coefficients of the series (3.21):

$$\begin{aligned} B_3 &= \frac{1}{\alpha} n_1 C_2 B_2 \\ B_4 &= \frac{1}{\alpha} [n_1 C_2 B_3 + n_1 C_3 B_2 + n_1^2 b^2 B_3^2] \\ &\cdots\cdots\cdots\cdots\cdots \\ B_m &= \frac{1}{\alpha} [n_1 C_2 B_{m-1} + \ldots + n_1 C_{m-1} B_2 + 2n_1^2 b^2 B_3 B_{m-1} + \ldots + n_1^2 b^2 B^2_{(m/2+1)}] \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{3.28}$$

The resulting series (3.17) is then obtained by multiplying the series (3.21) by $r^2$, and by setting $\mu = r$.

Comparing (3.16) and (3.18), we can establish the validity of the inequality (3.18), and from this follows the convergence of (2.7). Therefore, in the case under consideration, the Liapunov function $v(x)$ and the optimal control $u(x)$ do exist and are analytic functions in a neighborhood of the point $x = 0$.

**4.** One can give a formal method for the construction of a control $u = \{u_1, \ldots, u_n\}$, a vector function satisfying the condition of a minimum integral deviation of the system from a given motion. Let us consider a control system described by the differential equations

$$dx_i / dt = f_i(x, u) \qquad (i = 1, \ldots, n) \tag{4.1}$$

Suppose that along the trajectories of this system the following integral has a minimum:

$$\int_0^\infty G(x, u)\, dt = \min \tag{4.2}$$

Here $x = \{x_1, \ldots, x_n\}$ is an $n$-dimensional vector in the phase coordinates of the given system; $u = \{u_1, \ldots, u_n\}$ is a vector function which describes the control; $f_i(x, u)$ and $G(x, u)$ are analytic functions in the neighborhood of the origin

$$f_i(x, u) = f_i^{(1)}(x) + \sum_{k=1}^{n} b_{ik} u_k + \varphi_i(x, u) \qquad (i = 1, \ldots, n) \tag{4.3}$$

$$G(x, u) = \psi^{(2)}(x) + \sum_{i=1}^{n} C_i^{(1)}(x)\, u_i + \sum_{ik=1}^{n} d_{ik} u_i u_k + G_1(x, u) \tag{4.4}$$

where $\phi_i(x, u)$ and $G_1(x, u)$ are analytic functions that contain terms of higher order in $x$ and $u$; the coefficients $b_{ik}$ and $d_{ik}$ are such that the $n \times n$ matrices $B = \| b_{ik} \|_1^n$ and $D = \| d_{ik} \|_1^n$ are nonsingular.

Let $v(x)$ and $u(x) = \{u_1(x), \ldots, u_n(x)\}$ be functions satisfying the conditions (a) to (c) of Section 1, whereby the minimum in (c) is taken for all $u_k(x)$. Repeating all the considerations of that section, we obtain the following equations for the determination of $v(x)$ and $u(x)$:

$$\sum_{i=1}^{n} f_i(x, u) \frac{\partial v}{\partial x_i} + G(x, u) = 0 \tag{4.5}$$

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial u_k} \frac{\partial v}{\partial x_i} + \frac{\partial G}{\partial u_k} = 0 \qquad (k = 1, \ldots, n) \tag{4.6}$$

We will look for the solutions of the system of equations (4.5) and (4.6) in the form of series

$$v(x) = v^{(2)}(x) + \ldots + v^{(m)}(x) + \ldots \tag{4.7}$$

$$u_k(x) = u_k^{(1)}(x) + \ldots + u_k^{(m-1)}(x) + \ldots \qquad (k = 1, \ldots, n) \tag{4.8}$$

Let us substitute (4.3), (4.4), (4.7), and (4.8) into Equations (4.6) and (4.5), and equate to zero the coefficients of various powers of $x$.

Since the matrices $B$ and $D$ are nonsingular, the linear problem

$$\frac{dx_i}{dt} = f_i^{(1)}(x) + \sum_{k=1}^{n} b_{ik} u_k^{(1)} \qquad (i = 1, \ldots, n) \tag{4.9}$$

$$\int_0^\infty \left(\psi^{(2)}(x) + \sum_{i=1}^{n} C_i^{(1)}(x) u_i^{(1)} + \sum_{ik=1}^{n} d_{ik} u_i^{(1)} u_k^{(1)}\right) dt = \min \tag{4.10}$$

is solvable.

Liapunov's function $v^2(x)$ will be a positive-definite quadratic form, and the optimal control will satisfy the following system of linear nonhomogeneous equations:

$$2 \sum_{i=1}^{n} d_{ik} u_i^{(1)}(x) = -\sum_{i=1}^{n} b_{ik} \frac{\partial v^{(2)}}{\partial x_i} - C_k^{(1)}(x) \qquad (k = 1, \ldots, n) \tag{4.11}$$

Let us suppose that the functions $v^{(2)}(x), \ldots, v^{(m-1)}(x)$, and $u_k^{(1)}(x), \ldots, u_k^{(m-2)}(x)$ $(k = 1, \ldots, n)$ have been found. Let us write down the terms of the $m$th order of Equation (4.5), and the terms of the $(m - 1)$st order of Equation (4.6); we then obtain

$$\sum_{i=1}^{n} \left(f_i^{(1)}(x) + \sum_{k=1}^{n} b_{ik} u_k^{(1)}(x)\right) \frac{\partial v^{(m)}}{\partial x_i} + \sum_{k=1}^{n} u_k^{(m-1)}(x) \left(\sum_{i=1}^{n} b_{ik} \frac{\partial v^{(2)}}{\partial x_i} + C_k^{(1)}(x)\right) + $$
$$+ 2 \sum_{ik=1}^{n} d_{ik} u_i^{(1)}(x) u_k^{(m-1)}(x) = F_1^{(m)}(x) \tag{4.12}$$

$$\sum_{i=1}^{n} b_{ik} \frac{\partial v^{(m)}}{\partial x_i} + 2 \sum_{i=1}^{n} d_{ik} u_i^{(m-1)}(x) = F_k^{(m-1)}(x) \qquad (k - 1, \ldots, n) \tag{4.13}$$

where $F_1^{(m)}(x)$ and $F_k^{(m-1)}(x)$ are known forms.

With the aid of the system (4.11), Equation (4.12) can be reduced to the form

$$\left(\frac{dv^{(m)}}{dt}\right)_{(4.9)} = F_1^{(m)}(x) \tag{4.14}$$

Since the system of linear equations (4.9) is asymptotically stable, there exists a unique solution $v^{(m)}(x)$ of Equations (4.14).

Substituting the found value of the function $v^{(m)}(x)$ into (4.13), we obtain for the functions $u_k^{(m-1)}(x)$ a system of linear nonhomogeneous equations which also has a unique solution because its determinant is, by hypothesis, different from zero. Thus, one can successively determine the forms of any order in the series (4.7) and (4.8). Therefore, if the matrices $B$ and $D$ are nonsingular, then there exists a unique formal solution of the problem.

In conclusion, let us consider the following special case:

$$\frac{dx_i}{dt} = f_i(x) + \sum_{k=1}^{n} b_{ik} u_k \qquad (i = 1, \ldots, n) \tag{4.15}$$

$$\int_0^\infty \left( \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} u_i^2 \right) dt = \min \tag{4.16}$$

Equations for definition of $v(x)$ and $u(x)$ will have the form

$$\sum_{i=1}^{n} f_i(x) \frac{\partial v}{\partial x_i} - \frac{1}{4} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} b_{ik} \frac{\partial v}{\partial x_i} \right)^2 + \sum_{i=1}^{n} x_i^2 = 0 \tag{4.17}$$

$$u_k = -\frac{1}{2} \sum_{i=1}^{n} b_{ik} \frac{\partial v}{\partial x_i} \qquad (k = 1, \ldots, n) \tag{4.18}$$

Repeating the arguments of Section 3, one can show that Liapunov's function $v(x)$, the solution of Equation (4.17), is an analytic function. Therefore, the optimal control $u(x) = \{u_1(x), \ldots, u_n(x)\}$ exists and is an analytic function in the neighborhood of the point $x = 0$ if the matrix $B = \| b_{ik} \|_1^n$ is nonsingular.

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